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LETTER TO THE EDITOR

Similarity solutions of the Kadomtsev–Petviashvili equation

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Abstract. Using the direct method introduced by Clarkson and Kruskal (CK) recently, we obtain the similarity reductions of the Kadomtsev-Petviashvili equation (KPE). Some of them possess the same Painlevé properties as that of the Boussinesq equation (BE) obtained by CK. However, the similarity solutions of KPE may include some (more than three) arbitrary functions of time. There exist some other types of similarity reductions of the KPE different from the similarity reductions of the BE obtained by CK.

The standard method for finding similarity reductions of a given partial differential equation (PDE) is to use the method [1] (and/or the non-classical method [2]) of group-invariant solutions, which often involves a large amount of tedious algebra and auxiliary calculations. Recently, Clarkson and Kruskal (CK) presented a direct and simple method for the Boussinesq equation (BE) and other $(1+1)$-dimensional PDES [3]. In this letter, we apply the CK method to a $(2+1)$-dimensional model equation, the Kadomtsev-Petviashvili equation (KPE) [4]

$$-u_{tx}+6u_x^2+6uu_{xx}+u_{xxxx}+\gamma^2u_{yy}=0 \tag{1}$$

where γ is constant and subscripts denote differentiation.

Using the one-dimensional subalgebras of the infinite-dimensional Lie algebra of the KPE, David *et al* had reduced the KPE to some PDEs in two variables [5]; they are the BE, a once-differentiated Korteweg-de Vries equation (KdVE) and a linear equation. Here we would like to reduce the KPE to some ordinary differential equations (ODEs) by using CK's simple but powerful direct method rather than the classical Lie approach or non-classical symmetry reduction method.

All the similarity solutions of the form

$$u(x,y,t)=U(x,y,t,w(z)) \qquad z=z(x,y,t) \tag{2}$$

where U and z are functions of the indicated variables and $w(z)$ satisfies an ODE, may be obtained by substituting (2) into (1). However, as CK did for the BE, we can prove that it is also sufficient to seek a similarity reduction of the KPE in the special form

$$u(x,y,t)=\alpha(x,y,t)+\beta(x,y,t)w(z(x,y,t)) \tag{3}$$

rather than the most general form (2).

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Substituting (3) into (1) yields

$$\beta z_x^4 w'''' + (4\beta_x z_x^3 + 6z_x^2\beta z_{xx})w'''$$
$$+(-\beta z_t z_x + 6\alpha\beta z_x^2 + 6\beta_{xx}z_x^2 + 12\beta_x z_x z_{xx} + 4\beta z_x z_{xxx} + 3\beta z_{xx}^2 + \gamma^2\beta z_y^2)w''$$
$$+(-\beta_t z_x - \beta_x z_t - \beta z_{tx} + 12\alpha_x\beta z_x + 12\alpha\beta_x z_x + 6\alpha\beta z_{xx}$$
$$+4\beta_{xxx}z_x + 6\beta_{xx}z_{xx}4\beta_x z_{xxx} + \beta z_{xxxx} + 2\gamma^2\beta_y z_y + \gamma^2\beta Z_{yy})w''$$
$$+(-\beta_{tx} + 12\alpha_x\beta_x + 6\alpha\beta_{xx} + 6\beta\alpha_{xx} + \beta_{xxxx} + \gamma^2\beta_{yy})w + 6\beta^2 z_x^2(w')^2$$
$$+6\beta^2 z_x^2 ww'' + (6\beta_x^2 + 6\beta\beta_{xx})w^2 + (24\beta\beta_x z_x + 6\beta^2 z_{xx})ww'$$
$$+(-\alpha_{tx} + 6\alpha_x^2 + 6\alpha\alpha_{xx} + \alpha_{xxxx} + \gamma^2\alpha_{yy}) = 0 \tag{4}$$

where primes are z derivatives. Equation (1) is an ODE of $w(z)$ only for the ratios of coefficients of different derivatives and powers of $w(z)$ being functions of z. We use the coefficient of w'''' (i.e. βz_x^4) as the normalising coefficient and therefore we have:

$$6\beta^2 z_x^2 = \beta z_x^4 \Gamma_1(z) \tag{5}$$
$$4\beta_x z_x^3 + 6\beta z_x^2 z_{xx} = \beta z_x^4\Gamma_2(z) \tag{6}$$
$$-\beta z_t z_x + 6\alpha\beta z_x^2 + 6\beta_{xx}z_x^2 + 12\beta_x z_x z_{xx} + 4\beta z_x z_{xxx} + 3\beta z_{xx}^2 + \gamma^2\beta z_y^2 = \beta z_x^4\Gamma_3(z) \tag{7}$$
$$-\beta_t z_x - \beta_x z_t - \beta z_{tx} + 12\alpha_x\beta z_x + 12\alpha\beta_x z_x + 6\alpha\beta z_{xx} + 4\beta_{xxx}z_x + 6\beta_{xx}z_{xx}$$
$$+4\beta_x z_{xxx} + \beta z_{xxxx} + z\gamma^2\beta_y z_y + \gamma^2\beta z_{yy} = \beta z_x^4\Gamma_4(z) \tag{8}$$
$$-\beta_{tx} + 12\alpha_x\beta_x + 6\alpha\beta_{xx} + 6\alpha_{xx}\beta + \beta_{xxxx} + \gamma^2\beta_{yy} = \beta z_x^4\Gamma_\zeta(z) \tag{9}$$
$$-\alpha_{tx} + 6\alpha_x^2 + 6\alpha\alpha_{xx} + \alpha_{xxxx} + \gamma^2\alpha_{yy} = \beta z_x^4\Gamma_6(z) \tag{10}$$
$$6\beta_x^2 + 6\beta\beta_{xx} = \beta z_x^4\Gamma_7(z) \tag{11}$$
$$6\beta(4\beta_x z_x + \beta z_{xx}) = \beta z_x^4\Gamma_8(z) \tag{12}$$

where $\Gamma_1(z), \Gamma_2(z), \ldots, \Gamma_8(z)$ are functions of z to be determined.

Remark. There are three freedoms in the determination of α, β, z and w CK exploited:

(i) if $\alpha(x, y, t)$ has the form $\alpha = \alpha_0(x, y, t) + \beta(x, y, t)\Omega(z)$, then one can take $\Omega = 0$;

(ii) if $\beta(x, y, t)$ has the form $\beta = \beta_0(x, y, t)\Omega(z)$, then one can take Ω = constant; and

(iii) if $z(x, y, t)$ is determined by $\Omega(z) = z_0(x, y, t)$, where $\Omega(z)$ is any invertible function, then one can take $\Omega(z) = z$.

By using the remark, similar to CK's discussions for BE, we get the general solution of (5)–(12):

$$\Gamma_1(z) = 6 \tag{13}$$
$$\Gamma_2(z) = \Gamma_3(z) = \Gamma_7(z) = \Gamma_8(z) = 0 \tag{14}$$
$$\Gamma_4(z) = Az + B \tag{15}$$
$$\Gamma_5(z) = 2A \tag{16}$$
$$\Gamma_6(z) = -\tfrac{1}{3}[Az + B]^2 \tag{17}$$
$$z = x\theta(y, t) + \sigma(y, t) \tag{18}$$
$$\beta = z_x^2 = \theta^2 \tag{19}$$

and

$$\alpha = \frac{1}{6\theta^2}[x\theta_t\theta + \theta\sigma_t - \gamma\sigma_t - \gamma^2(x^2\theta_y^2 + \sigma_y^2 + 2x\theta_y\sigma_y)] \tag{20}$$

where A and B are arbitrary constants, θ and σ are only the functions of y and t which are determined by

$$\theta_{yy} = \frac{A}{\gamma^2}\theta^5 \tag{21}$$

and

$$-\theta_t + \gamma^2\sigma_{yy} = (A\sigma + B)\theta^4. \tag{22}$$

Substituting (13)-(18) into (4) leads to

$$w'''' + 6(w'^2 + ww'') + (Az + B)w' + 2Aw = \tfrac{1}{3}(Az + B)^2 \tag{23}$$

which is exactly same as that of CK's reduction equation for the BE after replacing $6w$ by w_1. This fact coincides with the fact that the KPE can be reduced to the BE by the classical Lie approach [5]. CK's discussions of the Painlevé property for the BE are valid here, i.e. (23) is equivalent to the fourth Painlevé equation; but, when $A = 0$, it is equivalent to the second Painlevé equation and, when $B = 0$ also, it is equivalent to either the first Painlevé equation or the Weierstrass elliptic function equation. Nevertheless, all the differences between the BE and the KPE appear in the constraints (20)-(22). Generally, four arbitrary functions of time t will be included in such types of the similarity solutions of the KPE because of equations (21) and (22) being PDEs.

Some special cases are listed here.

Case 1. $A = 0$, $B = 0$. In this case, the general solutions of (21) and (22) are

$$\theta(y,t) = a_1(t)y + a_0(t) \qquad \sigma(y,t) = \frac{1}{\gamma^2}[\tfrac{1}{6}a_{1t}(t)y^3 + \tfrac{1}{2}a_{0t}(t)y^2] + b_1(t)y + b_0(t) \tag{24}$$

and the similiarity reduction of the KPE is

$$\begin{aligned} u(x,y,t) = \frac{1}{6(a_1y + a_0)^2}\Bigg[&(a_1y + a_0)\left(xa_{1t}y + xa_{0t} + \frac{a_{1tt}}{6\gamma^2}y^3 + \frac{a_{0tt}}{2\gamma^2}y^2 + b_{1t}y + b_{0t}\right) \\ &- \gamma^2x^2a_1^2 - \frac{1}{\gamma^2}(\tfrac{1}{2}a_{1t}y^2 + a_{0t}y + \gamma^2b_1)^2 - 2xa_1(\tfrac{1}{2}a_1y^2 + a_0y + b_1\gamma^2)\Bigg] \\ &+ (a_1y + a_0)^2w(z) \end{aligned} \tag{25a}$$

$$z(x,y,t) = (a_1y + a_0)x + \frac{1}{\gamma^2}(\tfrac{1}{6}a_{tt}y^3 + \tfrac{1}{2}a_0y^2) + b_1(t)y + b_0(t) \tag{25b}$$

and

$$w'' + 3w^2 = c_1z + c_0 \tag{25c}$$

where c_1 and c_0 are arbitrary constants while a_1, a_0, b_1 and b_0 are all arbitrary functions of t. The travelling wave reduction arises as the special case of (25) when $a_{0t} = b_{1t} = b_{0t} = a_1 = 0$.

Case 2. $A=0$, $B\neq 0$. In this case, the general solution of (21) and (22) reads

$$\theta = a_1 y + a_0 \tag{26}$$

$$\sigma = \frac{B}{\gamma^2}\left[\tfrac{1}{30}a_1^4 y^6 + \tfrac{1}{5}a_0 a_1^3 y^5 + \tfrac{1}{2}a_0^2 a_1^2 y^4 + \tfrac{1}{6}(a_0^3 a_1 + a_{1t}/B)y^3 + \tfrac{1}{2}(a_0^4 + a_{0t}/B)y^2 + b_1 y + b_0\right]. \tag{27}$$

The similarity reduction of the KPE may be obtained by substituting (26) and (27) into (3), (18)-(20) and $w(z)$ satisfies

$$w''' + 6ww' + Bw = \tfrac{1}{3}B^2 z + c_0 \tag{28}$$

where c_0 is a constant while a_1, a_0, b_1 and b_0 are again all arbitrary functions of t.

Case 3a. $A\neq 0$. In this case we can always set $B=0$ in (22) (by substituting $\sigma\to\sigma - B/A$). Integrating (21) twice gives

$$y = \int^{\theta} \frac{d\theta_1}{(A\theta_1^6/(3\gamma^2) + a_1(t))^{1/2}} - a_0(t) \tag{29}$$

where $a_1(t)$ and $a_0(t)$ are still arbitrary functions of t.

Case 3b. $a_1(t)=0$. Equation (29) becomes

$$\theta(y,t) = \left(\frac{9\gamma^2}{4A}\right)^{1/4}(y + a_0(t))^{-1/2} \tag{30}$$

and the general solution of (22) is

$$\sigma(y,t) = b_1(t)(y + a_0(t))^{3/2} + b_0(t)(y+a_0(t))^{-1/2} + \frac{1}{2\gamma^2}\left(\frac{3\gamma^2}{4A}\right)^{1/2}(y+a_0(t))^{1/2} \tag{31}$$

with $b_1(t)$ and $b_0(t)$ being arbitrary functions of t. The corresponding similarity reduction can be obtained by substituting (30) and (31) into (3), (18)-(20).

Case 3c. $a_1(t)\neq 0$. In this case, (29) can be expressed explicitly in terms of some Jacobian elliptic functions. CK had given an example with a_1 and a_0 being constants and the parameters a_1 and A/γ^2 being complex, $a_1 = k^2 = \tfrac{1}{2}(1+i\sqrt{3})$ and $A/\gamma^2 = (k^2+1)/(3k^2)$, here we give some examples with real parameters or arbitrary functions of t.

(i) $A<0$, a_1 and a_0 are any functions of t:

$$\theta^{-2} = \frac{(1-k)^2}{2(1+6k+k^2)}\left(\frac{9A}{\gamma^2 a_1}\right)^{1/3} \times \phi_1^2\left[\frac{2\sqrt{2}}{(1+6k+k^2)^{1/2}}\left(\frac{-9a_1^2 A}{\gamma^2}\right)^{1/6}(y+a_0(t));\, k^2\right] + \left(\frac{-A}{3\gamma^2 a_1}\right)^{1/3} \tag{32a}$$

where ϕ_1 is

$$\phi_1(x;k^2) = \frac{(1-k)^{1/2}\,\mathrm{dn}(x;k^2)}{(1+k)^{1/2}(1+k\,\mathrm{sn}(x;k^2)) + [2k(1+\mathrm{sn}(x;k^2))(1+k\,\mathrm{sn}(x;k^2))]^{1/2}}. \tag{32b}$$

(ii) $A>0$, a_1 and a_0 are arbitrary functions of t:

$$\theta^{-2}=\frac{(1+k')^2}{2(1+k'^2-6k')}\left(\frac{9A}{\gamma^2 a_1}\right)^{1/3}$$
$$\times\phi_2^2\left[\frac{2\sqrt{2}}{(1-6k'+k'^2)^{1/2}}\left(\frac{9A_1^2A}{\gamma^2}\right)^{1/6}(y+a_0(t));\,k^2\right]-\left(\frac{A}{3\gamma^2 a_1}\right)^{1/3} \tag{33a}$$

$$\phi_2(x;\,k^2)=\frac{(1+k')\,\mathrm{sn}(x;\,k^2)}{[2(\mathrm{cn}^2(x;\,k^2)+\mathrm{dn}(x;\,k^2)+k'\,\mathrm{sn}^2(x;\,k^2))]^{1/2}}. \tag{33b}$$

The modulus k in (32) and/or the negative complementary modulus $-k'$ in (33) is a solution of the quartic equation

$$\chi^4+60\chi^3+134\chi^2+60\chi+1=0. \tag{34a}$$

One real solution of (34a) is

$$\chi\approx-0.017\,332\,38. \tag{34b}$$

The proofs of (32)-(34) may be obtained directly by the transformation

$$\theta^{-2}=\frac{-\lambda_0}{2\sigma_0}\left(\frac{9A}{\gamma^2 a_1}\right)^{1/3}\phi_0^2\left[\left(\frac{9a_1^2A}{-\sigma_0^3\gamma^2}\right)^{1/6}(y+a_0(t))\right]+\left(\frac{-A}{3\gamma^2 a_1}\right)^{1/3}$$

where $\phi_0(y_1)$ is a solution of the ϕ^4 model

$$\phi_{0y_1}^2=\sigma_0\phi_0^2+\tfrac{1}{2}\lambda_0\phi_0^4+\tfrac{2}{3}\sigma_0^2/\lambda_0$$

which possess various known special solutions such as ϕ_1 and ϕ_2 [6].

The correspondent function σ can be obtained by solving a linear differential equation (22) after substituting (32) or (33) into it.

Finally, it is necessary to point out that all the discussions above (and CK's results for BE) are valid only for $z_x\neq 0$. When $z_x=0$, (4) becomes

$$\gamma^2\beta z_y^2 w''+(-\beta_x z_t+2\gamma^2\beta_y z_y)w'+(-\beta_{tx}+12\alpha_x\beta_x+6\alpha\beta_{xx}+6\beta\alpha_{xx}+\beta_{xxxx}$$
$$+\gamma^2\beta_{yy})w+(6\beta_x^2+6\beta\beta_{xx})w^2-\alpha_{tx}+6\alpha_x^2+6\alpha\alpha_{xx}+\alpha_{xxxx}$$
$$+\gamma^2\alpha_{yy}=0. \tag{35}$$

There are two possibilities for (35) being an ODE of $w(z)$.

Case I. $z_y\neq 0$. In this case, the constraint conditions are

$$-\beta_x z_t+2\gamma^2\beta_y z_y+\gamma^2\beta z_{yy}=\gamma^2\beta z_y^2\Gamma_a(z) \tag{36}$$
$$-\beta_{tx}+12\alpha_x\beta_x+6\alpha\beta_{xx}+6\beta\alpha_{xx}+\beta_{xxxx}+\gamma^2\beta_{yy}=\gamma^2\beta z_y^2\Gamma_b(z) \tag{37}$$
$$6\beta_x^2+6\beta\beta_{xx}=\gamma^2\beta z_y^2\Gamma_c(z) \tag{38}$$
$$-\alpha_{tx}+6\alpha_x^2+6\alpha\alpha_{xx}+\alpha_{xxxx}+\gamma^2\alpha_{yy}=\gamma^2\beta z_y^2\Gamma_\alpha(z) \tag{39}$$

where Γ_a, Γ_b, Γ_c and Γ_α are some functions of z.

Case II. $z_y=0$. In this case, one can simply take $z=t$, the constraints read

$$-\beta_{tx}+12\alpha_x\beta_x+6\alpha\beta_{xx}+6\beta\alpha_{xx}-\beta_{xxxx}+\gamma^2\beta_{yy}=-\beta_x\Gamma_A(t) \tag{40}$$
$$6\beta_x^2+6\beta\beta_{xx}=-\beta_x\Gamma_B(t) \tag{41}$$
$$-\alpha_{tx}+6\alpha_x^2+6\alpha\alpha_{xx}+\alpha_{xxxx}+\gamma^2\alpha_{yy}=-\beta_x\Gamma_c(t) \tag{42}$$

with Γ_A, Γ_B and Γ_c being the undetermined functions of t.

It is clear that there exist many solutions to (36)-(39) and (40)-(42) which lead to some new similarity reductions of the KPE different from (23). Here is a special example:

$$\Gamma_a=\Gamma_c=0 \qquad \Gamma_d=-\Gamma_b=1 \tag{43}$$

$$z=\tfrac{3}{2}\ln(y+y_0(t)) \tag{44}$$

$$\beta=b_0(t)(y+y_0)^{1/2} \tag{45}$$

$$\alpha=-\frac{1}{6\gamma^2}(y+y_0)^{-2}\chi^2+\alpha_1(y,t)x+\alpha_0(y,t) \tag{46}$$

$$\alpha_1=B_0(t)(y+y_0)^{-2}+B_1(t)(y+y_0)^3-\tfrac{1}{6}y_{0t}(y+y_0)^{-1} \tag{47}$$

and

$$\begin{aligned}\alpha_0^2/\gamma^2=&\,C_1(t)(y+y_0)^2+C_2(t)(y+y_0)^{-1}-\gamma^{-2}b_0(t)(y+y_0)^{1/2}-\tfrac{3}{2}B_0^2(y+y_0)^{-2}\\&-\tfrac{1}{9}B_1^2(y+y_0)^8-3B_0B_1(y+y_0)^3+\tfrac{1}{2}B_1y_{0t}(y+y_0)^4\\&-\tfrac{1}{2}B_{0t}+\tfrac{1}{18}B_{1t}(y+y_0)^5+\tfrac{1}{12}y_{0tt}(y+y_0)\end{aligned} \tag{48}$$

where $b_0(t)$, $y_0(t)$, $B_0(t)$, $B_1(t)$, $C_1(t)$ and $C_2(t)$ are all functions of t.

The similarity reduction of the KPE is now

$$u=-\frac{1}{6\gamma^2}(y+y_0)^{-2}\chi^2+\alpha_1\chi+\alpha_0+\alpha_0+\beta(y,t)w(z) \tag{49}$$

and $w(z)$ is easy to get by solving

$$w''+w+1=0. \tag{50}$$

This is a new similarity reduction beyond (23).

In summary, we can reduce the KPE to some types of ODEs. Various arbitrary functions of t (more than three) have been included in our results. Generally, as CK pointed out, such types of reduction cannot be obtained by the classical Lie approach; however, for the first type of reduction (i.e. (23)), one can get it from combining the classical Lie approach and the non-classical symmetry reduction method. At first one can use the Lie approach to reduce the KPE to the BE [5], and then use the non-classical symmetry reduction method to reduce the BE to the ODE (23) [7]. The concrete forms of α, β and z obtained by this method may be different from our results but they are equivalent.

How to get all the solutions of (36)-(39) and (40)-(42), and whether there is any connection between these two types of reduction and the classical Lie approach or the non-classical symmetry reduction method, will be carefully discussed elsewhere.

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